Exercise 4

Solve the given ODEs:

$$y'' - 3y' + 2y = 0,\ y(0) = 2,\ y'(0) = 3$$

Solution

This is a linear ODE with two initial conditions, so the Laplace transform can be used to solve the problem. The Laplace transform of a function $f(x)$ is defined as

$$F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)\, dx,$$

so the derivatives of $f(x)$ transform as follows.

$$\mathcal{L}\{f'(x)\} = sF(s) - f(0)$$
$$\mathcal{L}\{f''(x)\} = s^2F(s) - sf(0) - f'(0)$$

Take the Laplace transform of both sides of the ODE.

$$\mathcal{L}\{y'' - 3y' + 2y\} = \mathcal{L}\{0\}$$

Use the fact that the operator is linear.

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = 0$$

Use the expressions above for the transforms of the derivatives and use the definition on the right side.

$$s^2Y(s) - sy(0) - y'(0) - 3[sY(s) - y(0)] + 2Y(s) = 0$$

Solve the equation for $Y(s)$.

$$(s^2 - 3s + 2)Y(s) + (-s + 3)y(0) - y'(0) = 0$$

$$(s^2 - 3s + 2)Y(s) = (s - 3)y(0) + y'(0)$$

$$Y(s) = \frac{(s - 3)y(0) + y'(0)}{s^2 - 3s + 2}$$

Factor the denominator and use the initial conditions, $y(0) = 2$ and $y'(0) = 3$.

$$Y(s) = \frac{2s - 3}{(s - 2)(s - 1)}$$

Use partial fraction decomposition to write the right side as a sum of simpler terms—ones that we know the inverse Laplace transforms of.

$$\frac{2s - 3}{(s - 2)(s - 1)} = \frac{A}{s - 2} + \frac{B}{s - 1}$$

Multiply both sides by the LCD, $(s - 2)(s - 1)$.

$$2s - 3 = A(s - 1) + B(s - 2)$$
$$2s - 3 = s(A + B) + (-A - 2B)$$

Comparing the coefficients on both sides, we obtain the following system of equations for A and B.

$$A + B = 2$$
$$-A - 2B = -3$$

The results are $A = 1$ and $B = 1$. Thus,

$$Y(s) = \frac{1}{s-2} + \frac{1}{s-1}.$$

Now that we have $Y(s)$, we can obtain $y(x)$ by taking the inverse Laplace transform of it.

$$\begin{aligned} y(x) &= \mathcal{L}^{-1}\{Y(s)\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-2} + \frac{1}{s-1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} \end{aligned}$$

Therefore,

$$y(x) = e^{2x} + e^{x}.$$